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Solidification in finite bodies with prescribed heat flux: bounds for the freezing time and removed energy

CH. CHARACH* and PETER B. KAHN†

* Applied Solar Calculations Unit, Jacob Blaustein Institute for Desert Research, Ben-Gurion University of the Negev, Sede Boqer Campus, 84990, Israel

† Department of Physics, State University of New York at Stony Brook, NY 11794, U.S.A.

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Abstract—One-dimensional, conduction-controlled solidification of initially overheated slabs, and cylindrical and spherical shells with insulated inner walls is considered. The heat flux from the outer face is taken to be constant or monotonically decreasing in time. We derive sufficiently simple and tight upper and lower bounds for the full freezing time and the extracted energy. These bounds are compared with some known approximate solutions, the accuracy of which is thereby established.

1. INTRODUCTION

HEAT conduction accompanied by melting or freezing is one of the most complicated topics in heat transfer. Only a few problems of this type admit exact analytic solutions [1]. More complicated problems are approached by numerical or approximate analytic methods, such as the heat balance integral method [2], the Megerlin method [3–5] or perturbation theory [6–8]. In many cases the accuracy of approximate solutions is unknown. Another approach, with a built-in criterion of maximal error, is the method of bounds [4, 9–13]. It focuses on development of reliable upper and lower bounds for the main dynamical parameters of the freezing or melting process. In refs. [11, 12] this method was applied to the analysis of one-dimensional melting in geometrically simple bodies with convective heating at the surface. It was assumed that the heat transfer takes place only in the solid phase. Solidification in such bodies with isothermal boundary conditions and an arbitrary level of initial overheating was considered recently in ref. [13].

Application of the method of bounds to one-dimensional, phase-change heat conduction with prescribed flux at the surface has been limited so far to planar geometry. Bolcy [9] considered the melting of a finite slab with an insulated back wall. He formulated a criterion which determines whether a solution of the problem with relaxed interface heat balance overpredicts (underpredicts) the interface location and the temperatures in the liquid and the solid. Unfortunately, the implementation of this criterion is very complicated if the slab is initially subcooled. Another approach to this problem was developed by Solomon [4] and Solomon *et al.* [10]. By ignoring the sensible heat in both phases they found an upper bound for the interface position. A lower bound for this parameter was obtained by replacing the actual temperature by that of a semi-infinite slab without a phase change, initially at the freezing temperature. The boundary condition for this slab was taken to be the same as for the actual finite slab. The bounds obtained this way are sufficiently tight only for very small fluxes. Glasser and Kern [11] derived bounds for the solidification time for a slab with prescribed constant fluxes both at the constant wall and at the moving interface. An analytic lower bound was derived by neglecting the sensible heat. The upper bound was found using the convexity of the solid temperature profile. The authors also proposed a way to improve the lower bound. However, it required a numerical solution of a nonlinear differential equation.

In the present paper the method of bounds is applied to the one-dimensional, conduction-controlled solidification of simple bodies with a given flux at the outer face. In addition to a slab with an insulated back wall we consider cylindrical and spherical shells, the inner surfaces of which are insulated. The heat flux from the outer surfaces is taken to be constant or monotonically decreasing in time. An arbitrary level of initial overheating is allowed. Using the total energy balance, the maximum principle [4, 14] and the integral representation of the solid temperature developed below, we formulate improved lower bounds for the full solidification time and the released energy. These bounds account for the sensible heat in both the solid and the liquid. The upper bounds are obtained using some theorems of the theory of superharmonic functions [14]. These upper bounds generalize those of Glasser and Kern to problems with time-dependent fluxes, initial overheating and cylindrical or spherical geometry. We present several examples that illustrate the accuracy of the bounds as compared with some approximations used in the earlier works.

NOMENCLATURE

a	ratio of the inner to outer radii of the body, R_2/R_1
c	heat capacity
$F=$	$Q(t)/S$ heat flux from the slab; $Q(t)/2\pi R$ heat flux from the cylindrical shell of unit length; $Q(t)/4\pi R^2$ heat flux from the spherical shell
k	thermal conductivity
L	latent heat of fusion
M_n	function defined by equation (12)
P_n	function defined by equation (12)
$Q(t)$	rate of energy extraction from the outer surface of the body
q	dimensionless flux, $FR_1/\alpha_l \rho L$
r	radial coordinate
R_1	outer radius of the body
R_2	inner radius of the body
S	area of the wall of a slab
t	time
t_1	moment at which the freezing begins
t^*	full solidification time
T	temperature
T_i	initial temperature
T_f	freezing point temperature.

Greek symbols

α	thermal diffusivity, $k/c\rho$
δ	phase-change front position
Δ	dimensionless phase-change front position, δ/R_1
ν	ratio of the thermal diffusivities, α_s/α_l
θ_l	dimensionless liquid temperature, $c_l(T_l-T_f)/L$
θ_s	dimensionless solid temperature, $c_s(T_s-T_f)/L$
θ_i	dimensionless initial temperature, $c_l(T_i-T_f)/L$
$\bar{\theta}_s$	function defined by equation (21)
ρ	density
τ	dimensionless time, $t\alpha_l/R_1^2$
τ_1	dimensionless time at which the freezing begins, $t_1\alpha_l/R_1^2$
τ^*	dimensionless full solidification time, $t^*\alpha_l/R_1^2$
ξ	dimensionless radial coordinate, r/R_1.

Subscripts

l	liquid
s	solid
n	geometrical index; $n = 0$, 1 and 2 correspond to the planar, cylindrical and spherical geometries, respectively.

2. FORMULATION OF THE PROBLEM AND INTEGRAL RELATIONS

We consider solidification in a slab, and a cylindrical or spherical shell. The back wall of the body is insulated, whereas the outer surface is subjected to a constant or monotonically decreasing outflux of heat (see Fig. 1). Initially the material is liquid at the temperature $T_l(0, r) = T_i(r)$, which is above or equal to the freezing temperature T_f. For $t > 0$ the heat is being extracted from the outer surface at a rate $F(t)$ per unit area (per unit length in the cylindrical case). At time $t = t_1$ the outer surface temperature becomes equal to the freezing temperature T_f. From then on the solidification begins and it is completed at $t = t^*$. We assume that the thermophysical parameters of each phase are constant and the densities of solid and liquid are equal $\rho_l = \rho_s = \rho$. We also assume that the heat transfer is only by one-dimensional heat conduction in the radial direction.

Using the dimensionless parameters and variables introduced in the Nomenclature the governing equations, and the initial and boundary conditions can be formulated as follows:

(a) The pre-solidification stage ($0 < \tau \leqslant \tau_1$)

$$(\partial\theta_l/\partial\tau) = \xi^{-n}\,\partial[\xi^n(\partial\theta_l/\partial\xi)]/\partial\xi, \quad a \leqslant \xi \leqslant 1 \quad (1)$$

$$(\partial\theta_l/\partial\xi)_{\xi=a} = 0, \quad (\partial\theta_l/\partial\xi)_{\xi=1} = -q(\tau), \quad \theta_l(0,\xi) = \theta_i(\xi). \quad (2)$$

Here $n = 0$, 1 or 2 correspond to the planar, cylindrical or spherical geometry, respectively. For

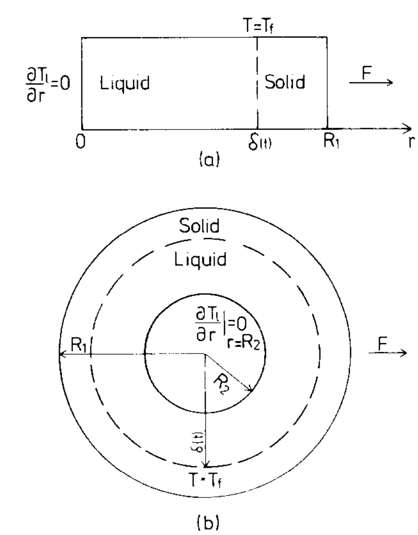

FIG. 1. (a) Solidification in a slab with an insulated wall. (b) Inward solidification in a cylindrical or spherical shell.

$n = 0$, a is zero. The function $q(\tau) > 0$ is presumed to be monotonically increasing or constant. The time τ_1 is defined by the requirement

$$\theta_l(\tau_1, 1) = 0. \quad (3)$$

(b) The solidification stage ($\tau_1 \leqslant \tau \leqslant \tau^*$)

$$(\partial\theta_l/\partial\tau) = \xi^{-n}\,\partial[\xi^n(\partial\theta_l/\partial\xi)]/\partial\xi, \quad a \leqslant \xi \leqslant \Delta \quad (4)$$

$$(\partial\theta_s/\partial\tau) = \nu\xi^{-n}\,\partial[\xi^n(\partial\theta_s/\partial\xi)]/\partial\xi, \quad \Delta \leqslant \xi \leqslant 1 \quad (5)$$

$$(\mathrm{d}\Delta/\mathrm{d}\tau) = \nu(\partial\theta_s/\partial\xi)_{\xi=\Delta} - (\partial\theta_l/\partial\xi)_{\xi=\Delta} \quad (6)$$

$$\theta_l(\tau, \Delta) = \theta_s(\tau, \Delta) = 0 \quad (7)$$

$$\Delta(\tau_1) = 1, \quad (\partial\theta_l/\partial\xi)_{\xi=a} = 0, \quad (\partial\theta_s/\partial\xi)_{\xi=1} = -q(\tau)/\nu. \quad (8)$$

Equations (4) and (5) describe the heat conduction in the liquid and solid phases. Equation (6) relates the liberated latent heat to the discontinuity of the heat flux at the moving interface, whereas equations (7) and (8) define other initial and boundary conditions.

Integrating equation (4) from $\xi = a$ to $\xi = \Delta$, equation (5) from $\xi = \Delta$ to $\xi = \xi$, adding the results, and using equation (6) one obtains

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_\Delta^\xi \xi'^n(\theta_s - 1)\,\mathrm{d}\xi' + \frac{\mathrm{d}}{\mathrm{d}\tau}\int_a^\Delta \xi'^n\theta_l\,\mathrm{d}\xi' = \nu\xi^n\,\partial\theta_s/\partial\xi, \quad \Delta \leqslant \xi \leqslant 1. \quad (9)$$

For $\xi = 1$, equations (8) and (9) lead to

$$\Delta^n(\mathrm{d}\Delta/\mathrm{d}\tau) = -q(\tau) - \int_a^\Delta \xi^n(\partial\theta_l/\partial\tau)\,\mathrm{d}\xi - \int_\Delta^1 \xi^n(\partial\theta_s/\partial\tau)\,\mathrm{d}\xi. \quad (10)$$

Integrating equation (9) from $\xi = \Delta$ to $\xi = \xi$ gives the following formal expression for θ_s:

$$\nu\theta_s = P_n(\xi, \Delta)(\mathrm{d}\Delta/\mathrm{d}\tau) + \int_\Delta^\xi P_n(\xi', \xi)(\partial\theta_s/\partial\tau)\,\mathrm{d}\xi' + M_n(\xi, \Delta)\int_a^\Delta \xi'^n(\partial\theta_l/\partial\tau)\,\mathrm{d}\xi', \quad (11)$$

$\Delta \leqslant \xi \leqslant 1$, where

$$P_n = \begin{cases} \xi - \xi', & n = 0 \\ \xi'\ln(\xi/\xi'), & n = 1 \\ \xi'(\xi - \xi')/\xi, & n = 2 \end{cases} \qquad M_n = \begin{cases} \xi - \Delta, & n = 0 \\ \ln(\xi/\Delta), & n = 1 \\ (\xi - \Delta)/\Delta\xi, & n = 2. \end{cases} \quad (12)$$

Integrating equation (10) with respect to time leads to

$$\int_{\tau_1}^\tau q(\tau')\,\mathrm{d}\tau' = [(1 - \Delta^{n+1})/(1 + n)] + \int_a^1 \xi^n\theta_l(\tau_1, \xi)\,\mathrm{d}\xi - \int_a^\Delta \xi^n\theta_l(\tau, \xi)\,\mathrm{d}\xi - \int_\Delta^1 \xi^n\theta_s(\tau, \xi)\,\mathrm{d}\xi. \quad (13)$$

Since $\Delta(\tau^*) = a$, equation (13) gives for $\tau = \tau^*$:

$$\int_{\tau_1}^{\tau^*} q(\tau')\,\mathrm{d}\tau' = [(1 - a^{n+1})/(n+1)] + \int_a^1 \xi^n\theta_l(\tau_1, \xi)\,\mathrm{d}\xi - \int_a^1 \xi^n\theta_s(\tau^*, \xi)\,\mathrm{d}\xi. \quad (14)$$

The LHS of this expression represents the energy extracted from the system during the solidification stage $\tau_1 \leqslant \tau \leqslant \tau^*$. According to equation (14) this energy is a sum of a latent heat, the residual sensible heat of liquid remaining from the pre-solidification regime, and the sensible heat of the solid released during the freezing of the body.

The heat balance for $0 \leqslant \tau \leqslant \tau_1$ is easily found by integrating equation (1) with the boundary conditions defined by equation (2):

$$\int_0^{\tau_1} q(\tau)\,\mathrm{d}\tau = \int_a^1 \xi^n[\theta_i(\xi) - \theta_l(\tau_1, \xi)]\,\mathrm{d}\xi. \quad (15)$$

Combining equations (14) and (15) one obtains the total energy extracted from the body during the time τ^*:

$$\int_0^{\tau^*} q(\tau)\,\mathrm{d}\tau = [(1 - a^{n+1})/(n+1)] + \int_a^1 \xi^n[\theta_i(\xi) - \theta_s(\tau^*, \xi)]\,\mathrm{d}\xi. \quad (16)$$

3. BOUNDS

We now develop upper and lower bounds for the full solidification time and the extracted energy using the integral relations derived in the previous section. Since the liquid temperature $\theta_l(\tau, \xi)$ is a non-negative decreasing function of time, it satisfies the following inequalities:

$$0 \leqslant \theta_l(\tau, \xi) \leqslant \theta_l(\tau_1, \xi), \quad \tau \geqslant \tau_1; \qquad \theta_l(\tau_1, \xi) \leqslant \theta_i. \quad (17)$$

The solid temperature θ_s is non-positive, $\theta_s \leqslant 0$. For monotonically nonincreasing fluxes $q(\tau^*) \geqslant q(\tau)$ and $(\partial\theta_s/\partial\tau) < 0$. Therefore, θ_s is a superharmonic function, i.e.

$$\xi^{-n}\,\partial[\xi^n(\partial\theta_s/\partial\xi)]/\partial\xi < 0. \quad (18)$$

According to the theory of such functions [14] $\theta_s > \bar{\theta}_s$, where $\bar{\theta}_s$ is the harmonic function defined by

$$\xi^{-n}\,\partial[\xi^n(\partial\bar{\theta}_s/\partial\xi)]/\partial\xi = 0 \quad (19)$$

$$\bar{\theta}_s(a) = 0, \quad (\partial\bar{\theta}_s/\partial\xi)_{\xi=1} = -q(\tau^*)/\nu. \quad (20)$$

The explicit form of $\bar{\theta}_s$ is given by

$$\bar{\theta}_s = \begin{cases} -q(\tau^*)\xi/\nu, & n = 0, \quad a = 0 \\ -q(\tau^*)[\ln(\xi/a)]/\nu, & n = 1, \quad a \neq 0 \\ -q(\tau^*)[(1/a) - (1/\xi)]/\nu, & n = 2, \quad a \neq 0. \end{cases} \quad (21)$$

Hence, the solid temperature is above the steady-state temperature $\bar{\theta}_s$:

$$\bar{\theta}_s < \theta_s(\tau, \xi) < 0, \quad \text{for } \Delta < \xi \leqslant 1, \quad \tau < \tau^*; \qquad \theta_s(\tau^*, a) = \bar{\theta}_s(a) = 0. \tag{22}$$

Using the inequalities (17) and (22) one obtains simple upper and lower bounds for the total energy released during the solidification process:

$$(1-a^{n+1})(1+n)^{-1} + \int_a^1 \xi^n \theta_i(\xi)\,\mathrm{d}\xi < \int_0^{\tau^*} q(\tau)\,\mathrm{d}\tau < (1-a^{n+1})(1+n)^{-1} + \int_a^1 \xi^n[\theta_i(\xi) - \bar{\theta}_s(\xi)]\,\mathrm{d}\xi. \tag{23}$$

The lower bound for the released energy in equation (23) neglects the sensible heat of the solid, whereas the upper bound overestimates the solid sensible heat by replacing the actual solid temperature by that of the steady-state problem, defined by equations (20) and (21). Since $q(\tau^*) > q(\tau)$, equation (23) also leads to the corresponding bounds for the total freezing time τ^*. In the particular case $n = 0$, $q =$ constant, and $\theta_i = 0$, the upper and lower bounds in equation (23) are identical to those of Glasser and Kern [11]. In the case $n = 0$ the lower bounds defined by equation (23) are also identical to those of refs. [4, 10].

The lower bounds for the released energy in equation (23) are obtained by substituting zero for the solid temperature in the heat balance equation (14). A better lower bound for the energy, which accounts for the solid sensible heat, can be developed using an improved upper bound for the solid temperature. Such a bound is suggested by equation (11). Since both $\partial\theta_s/\partial\tau$ and $\partial\theta_l/\partial\tau$ are negative, the latter equation implies the following inequality:

$$\theta_s < \nu^{-1}(\mathrm{d}\Delta/\mathrm{d}\tau)P_n(\xi, \Delta). \tag{24}$$

Its physical meaning is most easily understood in the planar case, $n = 0$, for which $\partial\theta_s/\partial\tau < 0$ implies $\partial^2\theta_s/\partial\xi^2 < 0$. Therefore, the temperature profile in the solid is a convex function of ξ with the maximal value $\theta_s = 0$ at $\xi = \Delta$. Consequently, the solid temperature is below the straight line given by

$$\theta_s < (\partial\theta_s/\partial\xi)_{\xi=\Delta}(\xi - \Delta) = (\partial\theta_s/\partial\xi)_{\xi=\Delta}P_0(\xi, \Delta), \quad 1 \geqslant \xi \geqslant \Delta. \tag{25}$$

Using the moving boundary condition (6) and the fact that both $(\mathrm{d}\Delta/\mathrm{d}\tau)$ and $(\partial\theta_l/\partial\xi)$ are negative, one obtains

$$\theta_s < [\nu^{-1}(\mathrm{d}\Delta/\mathrm{d}\tau) + (\partial\theta_l/\partial\xi)_{\xi=\Delta}]P_0(\xi, \Delta) < \nu^{-1}(\mathrm{d}\Delta/\mathrm{d}\tau)P_0(\xi, \Delta), \tag{26}$$

which is precisely the inequality (24). We now substitute instead of θ_s its upper limit (24) into the heat balance integral (13):

$$\int_{\tau_1}^{\tau} q(\tau')\,\mathrm{d}\tau' > (n+1)^{-1}(1-\Delta^{n+1}) + \int_a^{\Delta} \xi^n[\theta_l(\tau_1, \xi) - \theta_l(\tau, \xi)]\,\mathrm{d}\xi + \int_{\Delta}^1 \xi^n \theta_l(\tau_1, \xi)\,\mathrm{d}\xi - \nu^{-1}(\mathrm{d}\Delta/\mathrm{d}\tau) \times \int_{\Delta}^1 \xi^n P_n(\xi, \Delta)\,\mathrm{d}\xi. \tag{27}$$

The second term on the LHS of equation (27) represents the decrease of the sensible heat in that portion of the body which is still liquid at time τ. The third term is the released liquid sensible heat of the portion of the body which already solidified by the time τ. We can further strengthen the inequality (27) by neglecting the decrease of the temperature in the liquid portion of the body ($a \leqslant \xi \leqslant \Delta$):

$$\int_{\tau_1}^{\tau} q(\tau')\,\mathrm{d}\tau' > (n+1)^{-1}(1-\Delta^{n+1}) + \int_{\Delta}^1 \xi^n \theta_l(\tau_1, \xi)\,\mathrm{d}\xi - \nu^{-1}(\mathrm{d}\Delta/\mathrm{d}\tau)\int_{\Delta}^1 \xi^n P_n(\xi, \Delta)\,\mathrm{d}\xi. \tag{28}$$

This expression involves only one unknown function, the absolute value of the interface velocity $(-\mathrm{d}\Delta/\mathrm{d}\tau)$. The simple bounds for this quantity follow from equation (10):

$$0 < -\mathrm{d}\Delta/\mathrm{d}\tau < q(\tau)\Delta^{-n} \leqslant q(\tau^*)\Delta^{-n}. \tag{29}$$

The lower bound in equation (29), when substituted into (28), would lead to the same result as equation (23), whereas the upper bound would destroy the inequality (28). To overcome this difficulty we integrate equation (28) with respect to time from $\tau = \tau_1$ to $\tau = \tau^*$. This yields

$$\int_{\tau_1}^{\tau^*} \mathrm{d}\tau \int_{\tau_1}^{\tau} q(\tau')\,\mathrm{d}\tau' > -(n+1)^{-1}\int_a^1 (1-\Delta^{n+1})(\mathrm{d}\Delta/\mathrm{d}\tau)^{-1}\,\mathrm{d}\Delta - \int_a^1 (\mathrm{d}\Delta/\mathrm{d}\tau)^{-1}\,\mathrm{d}\Delta \int_{\Delta}^1 \xi^n \theta_l(\tau_1, \xi)\,\mathrm{d}\xi + \nu^{-1}\int_a^1 \mathrm{d}\Delta \int_{\Delta}^1 P_n(\xi, \Delta)\xi^n\,\mathrm{d}\xi. \tag{30}$$

This inequality involves the inverse of the interface velocity, and its structure is preserved when the upper bound of equation (29) is substituted into (30). This yields:

$$\int_{\tau_1}^{\tau^*} \mathrm{d}\tau \int_{\tau_1}^{\tau} q(\tau')\,\mathrm{d}\tau' > (1-a^{n+1})^2[2q(\tau^*)(n+1)^2]^{-1} + [q(\tau^*)]^{-1}\int_a^1 \Delta^n\,\mathrm{d}\Delta \int_{\Delta}^1 \xi^n \theta_l(\tau_1, \xi)\,\mathrm{d}\xi + \nu^{-1}\int_a^1 \mathrm{d}\Delta \int_{\Delta}^1 P_n(\xi, \Delta)\xi^n\,\mathrm{d}\xi. \tag{31}$$

Knowing the explicit expressions for $q(\tau)$ and $\theta_l(\tau_1, \xi)$, one can use the inequality (31) to derive lower bounds for the total freezing time and for the extracted energy. These bounds explicitly involve the contribution of the sensible heat of the solid, which is represented by the last term in equation (31). In the following section we illustrate application of the bounds (23) and (31) to several particular problems.

4. EXAMPLES

4.1. Slab with q = const, $\theta_i(\xi)$ = const

The upper and lower bounds corresponding to equation (23) are given by

$$[(1+\theta_i)/q] < \tau^* < (1+\theta_i)[1+q/2\nu(1+\theta_i)]/q. \quad (32)$$

These bounds are close to each other as long as the flux q is small, i.e. $q \ll 2(1+\theta_i)\nu$. The overheating tends to bring these limits closer together. In order to improve the lower bound we use equation (31). In the present case this gives

$$\tau^* > \tau_1 + q^{-1}\left[1+(q/3\nu)+2\int_0^1 \xi\theta_l(\tau_1,\xi)\,d\xi\right]^{1/2}. \quad (33)$$

Using the solution of the corresponding sensible heat problem for $\tau \leqslant \tau_1$

$$\theta_l(\tau,\xi) = \theta_i - q\tau - q\left\{[(3\xi^2-1)/6] - 2\pi^{-2} \times \sum_{n=1}^{\infty}(-1)^n n^{-2}[\exp(-\pi^2 n^2\tau)]\cos(\pi n\xi)\right\} \quad (34)$$

one obtains

$$\int_0^1 \xi\theta_l(\tau_1,\xi)\,d\xi = (q/3) - q\pi^{-2}\sum_{n=1}^{\infty} n^{-2}\exp(-\pi^2 n^2\tau) \times \{1-2\pi^{-2}n^{-2}[1-(-1)^n]\}. \quad (35)$$

The time, τ_1, is defined by the transcendental equation

$$\tau_1 = (\theta_i/q) - (1/3) + 2\pi^{-2}\sum_{n=1}^{\infty} n^{-2}\exp(-\pi^2 n^2\tau_1). \quad (36)$$

Notice that

$$(\theta_i/q) - (1/3) < \tau_1 \leqslant \theta_i/q, \quad 0 < \tau_1. \quad (37)$$

When $3\theta_i > q$, an analytic lower bound for τ^* can be derived combining equations (33) and (37):

$$\tau^* > (\theta_i/q) - (1/3) + q^{-1}[1+(q/3\nu)]^{1/2}. \quad (38)$$

In Table 1 we present the upper τ^{*+} and the lower τ^{*-} bounds for the full solidification time for the case $q = 1$, $\nu = 1$, $0 \leqslant \theta_i \leqslant 4$. This case cannot be treated by the perturbation theory because of the values of the parameters q and θ_i (the proper solidification time $\tau^* - \tau_1 \sim 1$, i.e. of the order of the heat diffusion time in each of the phases involved). The average of our bounds $\langle\tau^*\rangle$ predicts τ^* and the proper solidification time $\tau^* - \tau_1$ with an error not exceeding 13% for the range of parameters considered. As can be seen from Table 1, raising the initial temperature θ_i also leads to an increase of the pre-solidification time τ_1. However, the proper solidification time $\tau^* - \tau_1$ remains practically unchanged for $\theta_i \geqslant 1$. This effect can be understood by the fact that for $\theta_i \geqslant 1$ the transients of the liquid temperature are negligibly small at $\tau = \tau_1$. The liquid temperature at τ_1, $\theta_l(\tau_1, \xi)$ is practically the same for any initial overheating which satisfies the condition $\theta_i \geqslant 1$. The maximal value of $\theta_l(\tau_1, \xi)$ is at the insulated wall ($\xi = 0$). As can be seen from Table 1, for $\theta_i \geqslant 1$, $\theta_l(\tau = \tau_1, \xi = 0) = q/2 = 0.5$ [see equation (34)]. Consequently, the liquid resistance to the interface motion is the same for any $\theta_i \geqslant 1$, $q = 1$. According to the results of Table 1 the initial overheating can raise the proper solidification time $\tau^* - \tau_1$ by 22% as compared with that without overheating for the same heat outflux $q = 1$.

Table 1. Upper and lower bounds for the solidification time of initially overheated slab ($q = 1$, $\nu = 1$)

θ_i	0.00	0.33	0.50	1.00	2.00	3.00
τ_1	0.00	0.09	0.20	0.67	1.67	2.67
$\theta_l(\xi = 0, \tau = \tau_1)$	0.00	0.33	0.44	0.50	0.50	0.50
τ^{*+}	1.50	1.83	2.00	2.50	3.50	4.50
τ^{*-}	1.16	1.46	1.60	2.08	3.08	4.08
$\langle\tau^*\rangle$	1.33	1.65	1.80	2.29	3.29	4.29
$\langle\tau^*\rangle - \tau_1$	1.33	1.56	1.60	1.62	1.62	1.62
$\frac{\langle\tau^*\rangle - \tau^{*-}}{\langle\tau^*\rangle}$ %	13	12	11	9	6	5

The bounds become especially simple when $\theta_i = 0$ ($\tau_1 = 0$):

$$[1+(q/3\nu)]^{1/2}/q < \tau^* < (1/q) + (1/2\nu). \quad (39)$$

The case $\theta_i = 0$, q = const, was studied previously in refs. [2, 4, 11]. Using the bounds for the interface location given in ref. [4] one obtains the corresponding bounds for τ^*

$$(1/q) < \tau^* < (1/q) + (2/\pi\nu) + 2(\pi\nu)^{-1/2} \times [(1/\pi\nu) + (1/q)]^{1/2}. \quad (40)$$

The expressions for τ^* obtained by the Megerlin method [4] and by the heat balance integral approximation [2] are respectively given by

$$\tau^* = (1/2q) + \{[1+(4q/\nu)]^{3/2} - 1\}/12q^2 \quad (41)$$

$$\tau^* = \{5 + (q/\nu) + [1+(4q/\nu)]^{1/2}\}/6q. \quad (42)$$

In Fig. 2 we present the bounds (39), (40) and approximations (41), (42) for the case $\nu = 1$, $0 < q \leqslant 5$. As follows from Fig. 2 the bounds (39) are substantially more precise than those given by equation (40). The maximal error $(\tau^{*+} - \tau^{*-}/\tau^{*-})$ at $q = 0.5$ is now 15% instead of 98%, and 112% instead of 725% at $q = 5.0$. The average $\langle\tau^*\rangle$ of the bounds (39) is predicted with an error $\{\langle\tau^*\rangle - \tau^{*-}\}/\langle\tau^*\rangle$ varying from 7% at $q = 0.5$ to 43% at $q = 5.0$. This average is very close to the approximations (41) and (42). This result validates the above approximate solutions for fluxes with $q \gtrsim 1$, where the perturbation theory cannot be applied.

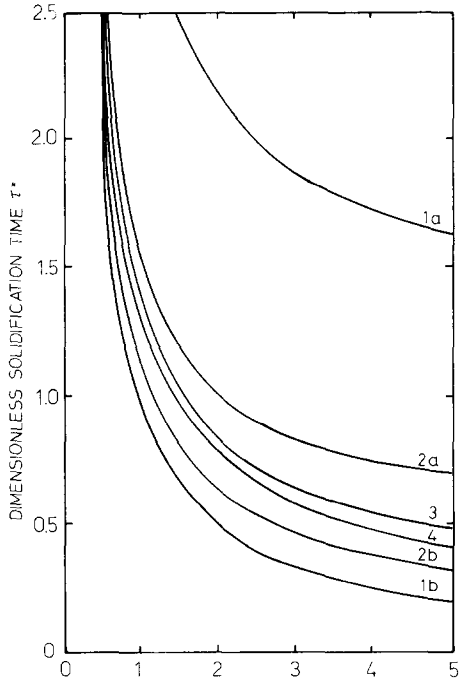

FIG. 2. Slab solidification time as a function of the heat flux ($v = 1$, $\theta_i = 0$, $q = \text{const}$). Curves 1a and 1b—the bounds corresponding to equation (40); curves 2a and 2b—the improved bounds, equation (39); curve 3—the heat balance method, equation (42); curve 4—the Megerlin method, equation (41).

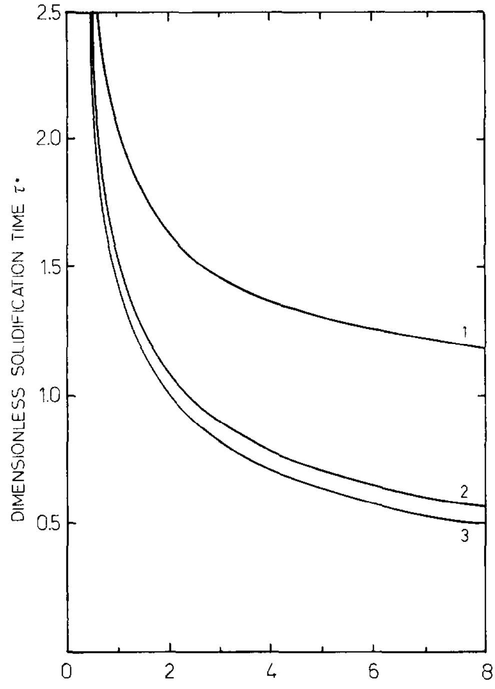

FIG. 3. Slab solidification time as a function of the flux parameter q_0 ($q = q_0\tau$, $\theta_i = 0$, $v = 1$). Curve 1—the upper bound in equation (41); curve 2—the improved lower bound, equation (44); curve 3—the lower bound in equation (43).

4.2. *Slab with* $q = q_0\tau$, $\theta_i = 0$

For this case equation (25) leads to the following bounds:

$$(2/q_0)^{1/2} < \tau^* < [2(1+q_0/8v^2)/q_0]^{1/2}+(2v)^{-1}. \quad (43)$$

An improved lower bound, which accounts for the sensible heat in the solid phase, is obtained using equation (31):

$$\tau^{*4} > (\tau^*/q_0 v)+(3/q_0^2). \quad (44)$$

The bounds corresponding to the particular case $v = 1$ are illustrated in Fig. 3. The average of the lower bound (44) and the upper bound in equation (43) predicts the full solidification time with an error varying from 13% to 55% in the range $0.5 \leqslant q_0 \leqslant 8.0$.

4.3. *Cylindrical shell* $\theta_i = 0$, $q = const$

The bounds corresponding to equation (25) are given by

$$[(1-a^2)/2q] \leqslant \tau^* \leqslant [(1-a^2)/2q] - [\ln a+(1-a^2)/2]/2v. \quad (45)$$

These bounds are close to each other as long as $(1-a^2) \gg q|[\ln a+(1-a)/2]|$. The accuracy of the upper bound decreases when a tends to zero. An improved lower bound corresponding to equation (33) is found to be

$$\tau^* \geqslant [(1-a^2)/2q]\{1+q[2a^2\ln a+(1-a^2)/2] \times [(1-a^2)^2 v]^{-1}\}^{1/2}. \quad (46)$$

The lower bound (46) is applicable even when $a = 0$. In Fig. 4 these bounds are shown for the case $v = 1$, $a = 0.4$. The maximal error of the improved bounds varies from 4% at $q = 0.1$ to 100% at $q = 3.0$. The Megerlin method applied to the present problem in ref. [5] leads to the following expression:

$$\tau^* = [(1-a^2)/4q]+\{1-a^2[1-4q(\ln a)/v]\}(4q)^{-1} + (\pi/32qv)^{1/2}[\exp(v/2q)] \times \{\text{erf}[(v/2q)-2\ln a]^{1/2}-\text{erf}(v/2q)^{1/2}\}. \quad (47)$$

In Fig. 4 we present the values of τ^* corresponding to equation (4) for $v = 1$, $a = 0.4$. These values of τ^* are almost the same as the improved lower bound given by equation (46).

5. CONCLUDING REMARKS

In this paper we develop lower and upper bounds for the solidification time and the energy extracted during freezing of slabs, and cylindrical and spherical shells with imposed heat flux at their outer surfaces. The inner surfaces of these bodies were assumed to be insulated, and the outfluxes of heat were taken to be constant or monotonically decreasing in time.

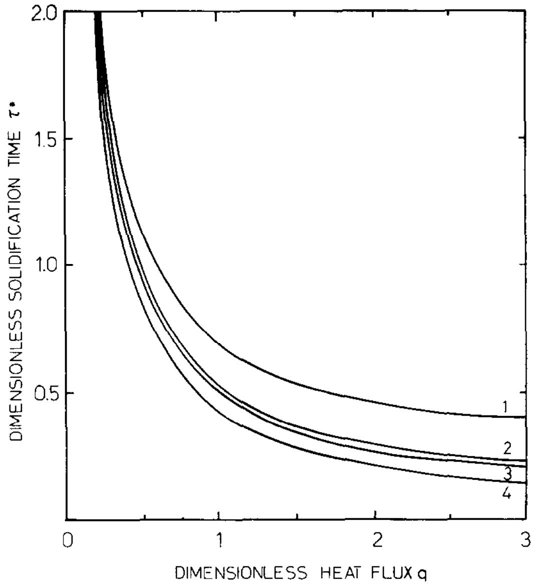

FIG. 4. Solidification time of the cylindrical shell as a function of the flux (q = const, $\theta_i = 0$, $v = 1$, $a = 0.4$). Curve 1—the upper bound in equation (45); curve 2—the approximate solution, equation (47); curve 3—the improved lower bound, equation (46); curve 4—the lower bound in equation (45).

The lower and upper bounds derived in the present work account for the initial overheating. By including the contributions of the sensible heat of the liquid and the solid, the lower bounds are substantially improved, as compared with the results of earlier studies. Our upper bounds account for the finite size of the systems considered and, therefore, are more accurate than those found in refs. [4, 10]. Due to these features the new bounds are sufficiently tight not only in the case of slow solidification, for which $q \ll 1$, but also for the $q \simeq 1$ case. The accuracy of the bounds increases as $q \to 0$.

For a slab with a constant flux and no initial overheating the maximal error of the new bounding solutions for τ^* is about 7 times lower than of those given in refs. [4, 10]. The new bounds also validate the approximate solutions obtained for the same problem by the heat balance integral method [2] and by the Megerlin method [4] for the range of fluxes $q \gtrsim 1$, where the perturbation theory is invalid. We also examined the accuracy of the bounds for the case of a substantially overheated slab. The overheating was shown to increase the time of the solidification stage by about 22% for $q = 1$ and $\theta_i \geqslant 1$. For inward solidification in a cylindrical shell the limits for τ^* derived in the present paper validate the approximate solution of ref. [5], which is slightly above our lower bound for τ^*. The method presented in this paper can be extended to the outward solidification in geometrically simple bodies, as well as to the melting of such bodies by a constant or monotonically increasing heat flux imposed at their surfaces.

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SOLIDIFICATION DE CORPS FINIS AVEC UN FLUX THERMIQUE PRESCRIT: LIMITES POUR LE TEMPS DE GEL ET POUR L'ENERGIE EXTRAITE

Résumé—On considère la solidification monodimensionnelle, contrôlée par la conduction, de plaques, de coques cylindriques ou sphériques initialement surchauffées. Le flux de chaleur enlevé sur la face externe est choisi constant ou monotoniquement décroissant dans le temps. On obtient des limites suffisamment simples, supérieures et inférieures pour le temps total de solidification et pour l'énergie extraite. Ces limites sont comparées avec quelques solutions approchées connues et dont la précision est connue.

ERSTARRUNG IN ENDLICHEN KÖRPERN MIT AUFGEPRÄGTER WÄRMESTROMDICHTE: GEFRIERZEIT UND ABGEFÜHRTE ENERGIE

Zusammenfassung—Es wurde die Erstarrung (im wesentlichen infolge eindimensionaler Wärmeleitung) von anfänglich überhitzten Platten sowie zylinder- und kugelförmigen Schalen mit wärmegedämmten Innenwänden betrachtet. Die Wärmestromdichte an der äußeren Oberfläche wurde als zeitlich konstant oder monoton abnehmend angenommen. Es wurden hinreichend einfache und dicht beieinanderliegende obere und untere Grenzen für die vollständige Gefrierzeit und die entzogene Energie hergeleitet. Diese Grenzen wurden mit einigen bekannten Näherungslösungen verglichen, deren Genauigkeit dabei bestätigt werden konnte.

ПРОЦЕСС ЗАТВЕРДЕВАНИЯ В ТЕЛАХ КОНЕЧНЫХ РАЗМЕРОВ ПРИ ЗАДАННОМ ТЕПЛОВОМ ПОТОКЕ: ОПРЕДЕЛЕНИЕ ПРЕДЕЛЬНЫХ ЗНАЧЕНИЙ ВРЕМЕНИ ЗАТВЕРДЕВАНИЯ И КОЛИЧЕСТВА ОТВЕДЕННОГО ТЕПЛА

Аннотация—Рассматривается одномерная задача затвердевания за счет отвода тепла теплопроводностью от перегретых в начальный момент времени пластин, а также цилиндрических и сферических оболочек с теплоизолированными внутренними стенками. Предполагается, что тепловой поток с наружной стороны поверхности не изменяется или монотонно уменьшается во времени. Получены довольно простые и компактные верхние и нижние пределы для полного времени затвердевания и количества отведенного тепла. Проведено сравнение с некоторыми известными приближенными решениями и тем самым установлена их точность.